Auditorium Exercise Sheet 5

Differential Equations I for Students of Engineering Sciences

Eleonora Ficola

Department of Mathematics of Hamburg University

Winter Semester 2023/2024

04.12.2023

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Linear homogeneous ODEs

Consider a linear, homogeneous ODE of order $m \in \mathbb{N}$

$$A_m(t)y^{(m)}(t) + \cdots + A_2(t)y''(t) + A_1(t)y'(t) + A_0(t)y(t) = 0, \qquad (1)$$

with coefficients $A_k \in \mathrm{C}(I)$.

- There are exactly m linearly independent solutions of (1).
- If $y_1, y_2, \ldots, y_m$ are m linearly independent solutions of (1), then they build a **basis** of the space of solutions of (1) and $M := \{y_1, \ldots, y_m\}$ defines a **fundamental system** of the ODE (1).
- The **general solution** of the homogeneous ODE (1) is given by

$$y_h(t) := C_1 y_1(t) + C_2 y_2(t) + \cdots + C_m y_m(t), \qquad \text{with } C_k \in \mathbb{R}.$$

- Question: how to find y_k?

Resolution of linear homogeneous ODEs with constant coefficients

In case the coefficients in (1) are constants, we get:

$$a_m y^{(m)}(t) + a_{m-1} y^{(m-1)}(t) + \cdots + a_2 y''(t) + a_1 y'(t) + a_0 y(t) = 0, \qquad (2)$$

for $a_k \in \mathbb{R}$.

- Define the **characteristic polynomial** of (2) as

$$P(\lambda) := a_m \lambda^m + a_{m-1}\lambda^{m-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0$$

- If λ is a root (zero) of P, then the function $e^{\lambda t}$ solves (2).
- In general, if λ is a root of P with (algebraic) multiplicity $d \in \mathbb{N}$, then

$$e^{\lambda t}, t \cdot e^{\lambda t}, \ldots, t^{d-1} \cdot e^{\lambda t}$$

are d linearly independent solutions of (2).

Example 1

Write a fundamental system and the general solution of the ODE

$$y^{(4)}(t) - 5y'''(t) + 6y''(t) + 4y'(t) - 8y(t) = 0.$$

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$$y^{(4)}(t) - 5y'''(t) + 6y''(t) + 4y'(t) - 8y(t) = 0.$$

Characteristic polynomial: $P(\lambda) = \lambda^4 - 5\lambda^3 + 6\lambda^2 + 4\lambda - 8 = (\lambda + 1)(\lambda - 2)^3$.

Roots of P are:

- $\lambda_1 = -1$, with multiplicity $d_1 = 1 \implies e^{-t}$ is a solution;
- $\lambda_2 = 2$, with multiplicity $d_2 = 3 \implies e^{2t}, te^{2t}, t^2e^{2t}$ are the other lin. indep. solutions.

Hence, a fundamental system is given by $M = \left\{e^{-t}, e^{2t}, te^{2t}, t^2e^{2t}\right\}$ and the general solution is

$$y_h(t) = C_1e^{-t} + C_2e^{2t} + C_3te^{2t} + C_4t^2e^{2t}, \qquad C_k \in \mathbb{R}.$$

Complex and real fundamental systems

- Recall: any polynomial of degree $m \in \mathbb{N}$ with real (or complex) coefficients has **exactly** m roots in $\mathbb{C}$ (counted with their multiplicity).

- If $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is a root of the characteristic polynomial P associated to (2) with **real** coefficients, then its complex conjugate $\overline{\lambda}$ is still root of P, since

$$P(\overline{\lambda}) = \sum_{k=0}^{m} a_k \overline{\lambda}^k = \overline{\sum_{k=0}^{m} a_k \lambda^k} \underset{\lambda \text{ root}}{=} \overline{0} = 0.$$

Meaning: complex solutions always appear in pairs of conjugates!

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  Meaning: complex solutions always appear in pairs of conjugates!

- **Example 2** The ODE $y'' - 2y' + 5y = 0$
  has characteristic polynomial $P(\lambda) = \lambda^2 - 2\lambda + 5$.
  Solve: $P(\lambda) = 0 \iff \lambda^2 - 2\lambda + 5 = 0 \iff \lambda = 1 \pm 2i$.

  Then a (complex) fundamental system is given by $\left\{e^{(1+2i)t}, e^{(1-2i)t}\right\}$ and the general solution is:
  $y_h(t) = C_1 e^{(1+2i)t} + C_2 e^{(1-2i)t} = C_1 e^t e^{2it} + C_2 e^t e^{-2it}$.

Complex and real fundamental systems

Euler formula: for $\theta \in \mathbb{R}$, it is

$$e^{\pm i\theta} = \cos(\theta) \pm i\sin(\theta).$$

If $\lambda = a + ib \in \mathbb{C}$ $(a, b \in \mathbb{R},\ b \neq 0)$, $e^{\lambda t} = e^{(a+ib)t} = e^{at}\cos(bt) + ie^{at}\sin(bt)$.
Let $\overline{\lambda} = a - ib$ be its complex conjugate.

$$\mathfrak{R}(e^{\lambda t}) = e^{at}\cos(bt) = \frac{e^{\lambda t} + e^{\overline{\lambda} t}}{2} \rightsquigarrow \text{ real part of } e^{\lambda t}$$

$$\mathfrak{I}(e^{\lambda t}) = e^{at}\sin(bt) = \frac{e^{\lambda t} - e^{\overline{\lambda} t}}{2i} \rightsquigarrow \text{ imaginary part of } e^{\lambda t}$$

- If λ is root of the characteristic polynomial P of (2) (and thus also $\overline{\lambda}$)
  $\Longrightarrow$ $e^{\lambda t}, e^{\overline{\lambda} t}$ are (complex, lin. indep.) solutions of (2)
  $\Longrightarrow$ $\mathfrak{R}(e^{\lambda t}), \mathfrak{I}(e^{\lambda t})$ are (real, lin. indep.) solutions of (2).

Example 3

Determine a **real** fundamental system for $y''(t) - 2y'(t) + 5y(t) = 0$.

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From Example 2 we know that $e^{(1+2i)t}$ and $e^{(1-2i)t}$ are 2 lin. indep. complex solutions.

Then $\mathfrak{R}(e^{(1+2i)t}) = e^t \cos(2t)$ and $\mathfrak{I}(e^{(1+2i)t}) = e^t \sin(2t)$ are lin. indep. real solutions.

A real fundamental system is then given by $\{e^t \cos(2t), e^t \sin(2t)\}$ and the corresponding general solution is

$$y_h(t) = C_1 e^t \cos(2t) + C_2 e^t \sin(2t).$$

Linear inhomogeneous ODEs

Consider a linear, inhomogeneous ODE of order $m \in \mathbb{N}$

$$A_m(t)y^{(m)}(t) + \cdots + A_2(t)y''(t) + A_1(t)y'(t) + A_0(t)y(t) = b(t), \quad (3)$$

with coefficients $A_k, b \in \mathrm{C}(I)$.

- If y_h is the general solution of the corresponding homogeneous equation and y_p is a particular solution of (3), then the general solution of (3) is given by

$$y(t) := y_h(t) + y_p(t)$$

- Question: how to determine y_p?

Resolution of linear inhomogeneous ODEs with constant coefficients by an ansatz

In case the coefficients in (3) are constant, we get:

$$a_m y^{(m)}(t) + a_{m-1} y^{(m-1)}(t) + \cdots + a_2 y''(t) + a_1 y'(t) + a_0 y(t) = b(t), \quad (4)$$

for $a_k \in \mathbb{R}$. If $b(t)$ is of "special" form, we can take an ansatz for y_p.

- If $b(t) = (b_0 + b_1 t + \cdots + b_q t^q) e^{\lambda t}$, we take the **exponential ansatz** y_p with:
  - If λ IS NOT a root of P, let $y_p(t) := (B_0 + B_1 t + \cdots + B_q t^q) e^{\lambda t}$;
  - If λ IS a root of P, let $y_p(t) := t^d (B_0 + B_1 t + \cdots + B_q t^q) e^{\lambda t}$ with d multiplicity of λ.
- If $b(t) = b_0 + b_1 t + \cdots + b_q t^q = (b_0 + b_1 t + \cdots + b_q t^q) e^{0t}$ **polynomial**, follow the previous case with $\lambda = 0$.

Resolution of linear inhomogeneous ODEs with constant coefficients by an ansatz

In case the coefficients in (3) are constant, we get:

$$a_m y^{(m)}(t) + a_{m-1} y^{(m-1)}(t) + \cdots + a_2 y''(t) + a_1 y'(t) + a_0 y(t) = b(t), \quad (4)$$

for $a_k \in \mathbb{R}$. If $b(t)$ is of "special" form, we can take an ansatz for y_p.

- If $b(t) = (b_0 + b_1 t + \cdots + b_q t^q)\cos(bt) + (c_0 + c_1 t + \cdots + c_q t^q)\sin(bt)$, we take the **trigonometric ansatz** y_p with:
  - If ib IS NOT a root of P, let $y_p(t) := (B_0 + B_1 t + \cdots + B_q t^q)\sin(bt) + (C_0 + C_1 t + \cdots + C_q t^q)\cos(bt)$;
  - If ib IS a root of P, let $y_p(t) := t^d(B_0 + B_1 t + \cdots + B_q t^q)\sin(bt) + t^d(C_0 + C_1 t + \cdots + C_q t^q)\cos(bt)$ with d multiplicity of λ.

Example 4

Determine the general solution of

$$y''(t) - 2y'(t) + 5y(t) = 13e^{-2t}. \tag{5}$$

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The general solution of (5) is given by $y(t) = y_h(t) + y_p(t)$, for y_h general solution of the corresponding homogeneous ODE

$$y''(t) - 2y'(t) + 5y(t) = 0. \tag{6}$$

In Example 3 we already computed $y_h(t) = C_1 e^t \cos(2t) + C_2 e^t \sin(2t)$.

Example 4

Determine the general solution of

$$y''(t) - 2y'(t) + 5y(t) = 13e^{-2t}. \tag{5}$$

The general solution of (5) is given by $y(t) = y_h(t) + y_p(t)$, for y_h general solution of the corresponding homogeneous ODE

$$y''(t) - 2y'(t) + 5y(t) = 0. \tag{6}$$

In Example 3 we already computed $y_h(t) = C_1 e^t \cos(2t) + C_2 e^t \sin(2t)$.

The inhomogeneity term is $b(t) := 13e^{-2t}$, with $\lambda = -2$ NO root of $P \implies$ take the exponential ansatz $y_p(t) := Ce^{-2t}$. Substitute into (5) to find C:

$$(4C + (-2)(-2)C + 5C)e^{-2t} = 13e^{-2t} \implies C = 1 \implies y_p(t) = e^{-2t}.$$

The general solution of (5) is:

$$y(t) = y_h(t) + y_p(t) = C_1 e^t \cos(2t) + C_2 e^t \sin(2t) + e^{-2t}.$$

Remark: linear combination of particular solutions

Suppose $y_{p,1}$ is a solution of the linear, inhomogeneous ODE

$$\sum_{k=0}^{m} A_k(t)y^{(k)}(t) = b_1(t)$$

and $y_{p,2}$ solves

$$\sum_{k=0}^{m} A_k(t)y^{(k)}(t) = b_2(t).$$

Then each linear combination $y(t) := \alpha y_{p,1} + \beta y_{p,2}$ $(\alpha, \beta \in \mathbb{R})$ solves

$$\sum_{k=0}^{m} A_k(t)y^{(k)}(t) = \alpha b_1(t) + \beta b_2(t).$$

Exercise 1

Determine a real fundamental system and the general solution of the following linear homogeneous ODE:

$$y^{(5)} - 4y^{(4)} + 9y''' - 18y'' + 20y' - 8y = 0.$$

Exercise 2

Determine the general solution of the differential equation

$$y'''(t) + y''(t) - 2y(t) = b_k(t)$$

in each of the following cases:

(i) $b_1(t) = e^{-t}$;

(ii) $b_2(t) = 2te^{t}$;

(iii) $b_3(t) = t^2 + 3$;

(iv) $b_4(t) = 25\cos(2t)$;

(v) $b_5(t) = -2t^2 + 4e^{-t} - 6$. Hint: Notice that $b_5(t) = 4b_1(t) - 2b_3(t)$.

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EXERCISE 2

(1) $\underbrace{y''' + y'' - 2y = b_k(t)}_{\text{3rd order linear ODE}}$, $t \in \mathbb{R}$

inhom. with constant coefficients

(i) $b_1(t) = e^{-t}$

General sol. of (1) is $y(t) = y_h(t) + y_p(t)$

- Determine y_h gen. sol. of $\left[y''' + y'' - 2y = 0 \right]$ (1_h).

Characteristic polynomial $P(\lambda) := \lambda^3 + \lambda^2 - 2 = (\lambda - 1)(\lambda^2 + 2\lambda + 2) =$

$= (\lambda - 1)(\lambda - 1 + i)(\lambda - 1 - i)$

$P(1) = 1 + 1 - 2$ ✓ $\rightarrow$ $\lambda_1 = 1$ root

λ^3	λ^2	0	-2	$\lambda - 1$
$-\lambda^3$	λ^2			$\lambda^2 + 2\lambda + 2$
	$2\lambda^2$	0	-2	
	$-2\lambda^2$	$+2\lambda$		
		2λ	-2	

$\hookrightarrow \Delta = 4 - 4 \cdot 2 = -4 = 4i^2$

$\lambda_{2/3} = \frac{-2 \pm \sqrt{4i^2}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$

ALGEBRAIC MULTIPLICITIES

- $\lambda_1 = 1$, $\boxed{d_1 = 1}$ root $\Rightarrow$ $e^{\lambda_1 t} \sim e^{t}$ sol. of (1_h)
- $\lambda_2 = -1 + i$, $\boxed{d_2 = 1}$ root $\Rightarrow$ $e^{\lambda_2 t} \sim e^{(-1+i)t} = e^{-t} e^{it} = e^{-t}(\cos(t) + i \sin(t)) = e^{-t}\cos(t) + i e^{-t}\sin(t)$ sol. of (1_h)
- $\lambda_3 = -1 - i \ (= \overline{\lambda_2})$, $\boxed{d_3 = 1}$ root $\Rightarrow$ $e^{\lambda_3 t} \sim e^{(1-i)t}$ sol. of (1_h)

A complex fund. system of (1_h): $M_{\mathbb{C}} = \{ e^{t}, e^{(-1+i)t}, e^{(-1-i)t} \}$ $\leftarrow$ 3 linearly independent sol.

A real fund. system of (1_h): $M_{\mathbb{R}} = \{ e^{t}, e^{-t}\cos(t), e^{-t}\sin(t) \}$

$\mathcal{R}(e^{\lambda_2 t}) = e^{-t} \cdot \cos(t)$

$\mathcal{I}(e^{\lambda_2 t}) = e^{-t} \sin(t)$

$y_h(t) = C_1 \cdot e^{t} + C_2 \cdot e^{-t}\cos(t) + C_3 e^{-t} \sin(t)$, $C_i \in \mathbb{R}$ $\leftarrow$ general solution of the corresponding homogeneous ODE

• Determine a particular sol. y_p of (1)

$b_1(t) = e^{-t} \rightsquigarrow e^{-1t}$ $\lambda = -1 \to$ NOT a root of P!

$b_1 = \{\text{pol. of deg. } 0\} \cdot e^{-t} \Rightarrow y_p = \{\text{pol. of deg. } 0\} e^{-t} = c\, e^{-t}$, we need to find c:

Differentiate y_p: $y_p' = -c e^{-t}$, $y_p'' = c e^{-t}$, $y_p''' = -c e^{-t}$

Impose: $\left[y_p''' + y_p'' - 2y_p = e^{-t} \right]$

$(-c + c - 2c) e^{-t} = e^{-t}$

$-2c = 1 \Rightarrow c = -\frac{1}{2} \rightsquigarrow$ $y_p(t) = -\frac{e^{-t}}{2}$

• Gen. sol. of (1) (i) is: $y(t) = y_h + y_p = c_1 e^t + c_2 e^{-t}\cos(t) + c_3 e^{-t}\sin(t) - \frac{e^{-t}}{2}$

(ii) $b_2(t) = 2t \cdot e^t \rightsquigarrow$ find y_p

$b_2 = \{\text{pol. of deg. } 1\} \cdot e^t$, e^{1t} with $\lambda = 1 \rightsquigarrow$ ROOT of P with mult. 1 $\Rightarrow$

$\Rightarrow y_p(t) = t^1 \cdot \{\text{pol. of deg. } 1\} e^t = t(at + b)e^t = (at^2 + bt)e^t$

a, b to be determined such that y_p is a solution

$y_p' = (2at + b + at^2 + bt) e^t$

$y_p'' = (2a + 2at + b + 2at + b + at^2 + bt)e^t = (2a + 2b + 4at + at^2 + bt)e^t$

$y_p''' = (4a + 2at + b + 2a + 2b + 4at + at^2 + bt)e^t = (6a + 6at + 3b + at^2 + bt)e^t$

Set: $y_p''' + y_p'' - 2y_p = 2te^t \Leftrightarrow (6a + 6at + 3b + at^2 + bt + 2a + 2b + 4at + at^2 + bt - 2at^2 - 2bt)e^t = 2te^t$

$\Leftrightarrow t(6a + 4a) + (6a + 3b + 2a + 2b) = 2t, \quad \forall t \in \mathbb{R}$

$10a\, t + (8a + 5b) = 2t + 0 \longrightarrow$ solve by comparison of the coefficients:

$\begin{cases} 10a = 2 \to a = 1/5 \\ 8a + 5b = 0 \to b = -\frac{8a}{5} = \frac{-8}{25} \end{cases}$ $\rightsquigarrow y_p(t) = \left(\frac{t^2}{5} - \frac{8t}{25}\right)e^t$

↓ general sol. of (1) (ii) is:

$y(t) = c_1 e^t + c_2 e^{-t}\cos(t) + c_3 e^{-t}\sin(t) + \left(\frac{t^2}{5} - \frac{8t}{25}\right)e^t$, $c_i \in \mathbb{R}$